Left braces and solutions of the quantum Yang-Baxter equation

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Introduction

The quantum Yang-Baxter Equation (YBE) is an important equation in mathematical physics. One of the fundamental open problems is to find all the solutions of the YBE. In last years, the involutive and non-degenerate solutions have received a lot of attention.

Given a non-empty set X, a map $r\colon X \times X \longrightarrow X \times X$ is a *set-theoretic solution* of the YBE if

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where the maps r_{12}, $r_{23}\colon X \times X \times X \longrightarrow X \times X \times X$ are defined as $r_{12} = r \times \mathrm{id}_X$, $r_{23} = \mathrm{id}_X \times r$. For all x, $y \in X$, we define two maps $f_x\colon X \longrightarrow X$ and $g_y\colon X \longrightarrow X$ by setting $r(x, y) = (f_x(y), g_y(x))$. The solution (X, r) is called *involutive* if $r^2 = \mathrm{id}_{X^2}$ and *non-degenerate* if f_x, g_y are bijective maps for all x, $y \in X$. By a solution of the YBE we mean a non-degenerate involutive set-theoretic solution, as in [3, 2].

The solutions of the YBE can be studied using group theory by considering two fundamental groups: the structure group $G(X, r) = \langle X \mid xy = f_x(y)g_y(x) \text{ for all } x, y \in X\rangle$, and the permutation group $\mathcal{G}(X, r) = \langle f_x \mid x \in X\rangle \leqslant \mathrm{Sym}(X)$.

On the other hand, Rump introduced in [5] a new algebraic structure, called *left brace*, to study the solutions of the YBE. A set B with two binary operations, $+$ and $\cdot$, is a left brace if $(B, +)$ is an abelian group, $(B, \cdot)$ is a group and

$$a \cdot (b + c) = a \cdot b + a \cdot c - a \quad \text{for all } a, b, c \in B.$$

If (X, r) is a solution of the YBE, then $G(X, r)$ and $\mathcal{G}(X, r)$ are left braces (see [3, Section 3]). Moreover, the results of [1] show that every finite left brace is isomorphic to the left brace $\mathcal{G}(X, r)$ for some finite solution of the YBE, and allow us to conclude that the problem of constructing all finite solutions of the YBE is reduced to describing all finite left braces.

We will introduce and analyse two brace theoretical properties that extend left and right nilpotency, and we will apply them to the study of the solutions of the YBE. *The word "brace" always means "left brace".*

Previous definitions

Definition 1 . Let $*$ be the binary operation on B defined by setting

$$a * b = a \cdot b - a - b,$$

for a, $b \in B$. If X, Y are subsets of the brace B, then

$$X * Y = \langle x * y \mid x \in X, y \in Y\rangle_+$$

and we can define inductively:

$$\mathrm{L}_0(X,Y) = Y;\ \ \mathrm{L}_n(X,Y) = X * \mathrm{L}_{n-1}(X,Y) \quad (n \geq 1);$$
$$\mathrm{R}_0(X,Y) = X;\ \ \mathrm{R}_n(X,Y) = \mathrm{R}_{n-1}(X,Y) * Y \quad (n \geq 1).$$

Definition 2 ([5]) . Given a brace B, its socle series is defined by setting $\mathrm{Soc}_0(B) = 0$, $\mathrm{Soc}(B) = \mathrm{Soc}_1(B)$, and

$$\mathrm{Soc}_{n+1}(B) := \{x \in B \mid x * a \in \mathrm{Soc}_n(B), \text{for all } a \in B\},$$

for all $n \geq 1$.

Definition 3 ([4]) . Let (X, r) be a solution of the YBE and assume that $r(x, y) = (f_x(y), g_y(x))$ for all x, $y \in X$. The *retraction* relation $\sim$ on X with respect to r is an equivalence relation defined by $x \sim y$ if $f_x = f_y$.
The natural induced solution $\mathrm{Ret}(X, r) = (X/{\sim}, \widetilde{r})$ is called the *retraction* of (X, r), and $\widetilde{r}$ is defined by

$$\widetilde{r}([x], [y]) = ([f_x(y)], [g_y(x)]) \quad \text{for all } [x],\ [y] \in X/{\sim}.$$

Define $\mathrm{Ret}^0(X, r) = (X, r)$, $\mathrm{Ret}^1(X, r) = \mathrm{Ret}(X, r)$ and

$$\mathrm{Ret}^m(X, r) = \mathrm{Ret}(\mathrm{Ret}^{m-1}(X, r)), m \geq 2.$$

Then (X, r) is said to be a *multipermutation solution of level m* if m is the smallest non-negative integer such that $|\mathrm{Ret}^m(X, r)| = 1$.

Left p-nilpotent braces

Definition 4 . Let B be a finite brace and let p be a prime.

(a) B is called *left nilpotent* if $\mathrm{L}_n(B, B) = 0$ for some $n \in \mathbb{N}$.

(b) B is called *left p-nilpotent* if $\mathrm{L}_n(B, B_p) = 0$ for some $n \in \mathbb{N}$, where B_p is the Sylow p-subgroup of the additive group of B.

The following two results show that left nilpotency of finite braces is a local property and characterise left p-nilpotent finite braces, respectively. Note that the second one is an extension of [6, Theorem 1] and also provides an alternative shorter proof of that result.

Lemma 5 . *Let B be a finite brace. Then B is left nilpotent if and only if B is left p-nilpotent for all primes p.*

Theorem 6 . *Let $(B, +, \cdot)$ be a finite brace and let p be a prime. Assume that $B_{p'}$, B_p are the Hall p'-subgroup and Sylow p-subgroup of the group $(B, +)$, respectively. Then the following statements are pairwise equivalent:*

1. *B is a left p-nilpotent brace.*
2. *$B_{p'} * B_p = 0$.*
3. *$B_{p'} * \Omega((B_p, +)) = 0$, where $\Omega((B_p, +))$ is the group generated by all element of order p in $(B_p, +)$.*
4. *The multiplicative group $(B, \cdot)$ is p-nilpotent.*

Corollary 7 ([6]) . *Let $(B, +, \cdot)$ be a finite brace. Then B is left nilpotent if and only if the multiplicative group $(B, \cdot)$ is nilpotent.*

Another application of left p-nilpotent finite braces is the following non-simplicity criterion.

Corollary 8 . *Let $(B, +, \cdot)$ be a finite brace and p be the smallest prime dividing the order of B. If the Sylow p-subgroups of $(B, \cdot)$ are cyclic, then B is left p-nilpotent. In particular, B is not simple if $|B| \neq p$.*

Right p-nilpotent braces

Definition 9 . Let B be a finite brace and p be a prime.

(a) B is called *right nilpotent* if $\mathrm{R}_n(B, B) = 0$ for some $n \in \mathbb{N}$.

(b) B is called *right p-nilpotent* if $\mathrm{R}_n(B_p, B) = 0$ for some $n \in \mathbb{N}$, where B_p is the Sylow p-subgroup of the additive group of B.

Theorem 10 . *Let B be a finite brace. Assume that the multiplicative group of B has an abelian normal Sylow p-subgroup for some prime p. Then B is right p-nilpotent.*

The following characterisation theorem includes an extension of [2, Proposition 6] which states that a brace B is right nilpotent if and only if the solution of the YBE associated to B is a multipermutation solution.

Theorem 11 . *Let B be a finite brace and let (B, r) be the solution of the YBE associated to B. Assume that p is a prime and B_p is the Sylow p-subgroup of $(B, +)$. Then the following statements are pairwise equivalent:*

1. *B is right p-nilpotent.*
2. *$B_p \subseteq \mathrm{Soc}_n(B)$ for some $n \geq 0$.*
3. *There exists some $n \geq 0$ such that the cardinality of $\mathrm{Ret}^n(B, r)$ is a p'-number.*

Proposition 12 . *Let B be a finite brace. Then B is right nilpotent if and only if B is right p-nilpotent for all primes p. Thus, right p-nilpotency is a local property.*

Recall a finite group is called *A-group* if all its Sylow subgroups are abelian, and is said to satisfy the *Sylow tower property* if G has a normal Hall π_p-subgroup for each prime p, where $\pi_p = \{q \mid p \leq q\}$.

Corollary 13 . *Let B be a finite brace. Assume that $(B, \cdot)$ an A-group with the Sylow tower property. Then B is right nilpotent.*

Finally, the following corollary is a significant improvement of [3, Theorem 3].

Corollary 14 . *Let (X, r) be a finite solution of the YBE. Assume that $\mathcal{G}(X, r)$ is an A-group with the Sylow tower property. Then (X, r) is a multipermutation solution.*

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Acknowledgements This work has been supported by the research grant MTM2014-54707-C3-1-P from the *Ministerio de Economía y Competitividad*, Spanish Government, and FEDER, European Union, and PROMETEO/2017/057 from *Generalitat* (Valencian Community, Spain). The third author has been supported by the predoctoral grant Atracció de Talent of Universitat de València. The fourth author has been supported by the grant number 201606890006 of the China Scholarship Council.